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SYNTHESIS OF A GENERATING SYSTEM IN THE PROBLEM OF CONTROLLING AN ELASTIC ROD†

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The controlled motions of an elastic rod are investigated using the example of a discrete model. The model consists of an arbitrary finite number of absolutely rigid links with elastic connections between them. For sufficiently high stiffness, the motions of an elastic system contain different frequency components: the fast motions of the elastic part and the slow motions of the system as a solid body. The motion of such systems is described by singularly perturbed equations. On changing to the standard form of Poincaré systems with a small parameter on the right-hand side of the equations of motion, the generating system which describes the motion of the elastic part may turn out to be conservative and the vibrational motions of the elastic part will be preserved in the system. A procedure for synthesizing the equations is proposed which enables one to form a generating system with the required properties. The effect which arises is of a substantially non-linear form. A theorem on the asymptotic stability of the steady state of the generating system as a whole is proved.

1. FORMULATION OF THE PROBLEM

As the object being controlled, we consider an elastic rod which can rotate about a fixed axis under the action of a controlling moment $M_1(t)$ which is applied at a fixed joint. In considering the effect of elastic forces on the controlled motions of the rod we confine ourselves to the case when there are no external forces apart from the controlling forces. This system can serve, in particular, as a model for investigating the effect of elasticity on the motion of manipulators with flexible links. The controlled motions will be investigated using the example of a discrete model of an elastic rod with an arbitrary finite number of degrees of freedom. This enables us to establish common features and to point to possible generalizations.

As the discrete model of an elastic rod of length l and mass m, let us consider a planar chain of an arbitrary finite number n of absolutely rigid links of length l_i and mass m_i $(l_1+\ldots+l_n=l$, $m_1+\ldots+m_n=m)$ connected to one another by means of cylindrical joints. We shall characterize the elasticity in the jth $(j=2,\ldots,n)$ connection by the restoring moment $Q_j=-\bar{c}_jq_j$, where q_j is the angle between the links with numbers $j-1$ and J and $\bar{c}_j$ are the stiffness coefficients. We denote by q_1 the angle which the first link makes with a certain fixed axis lying in the plane of rotation of the links. Since this system is regarded as a dynamic model of an elastic rod, the quantities $\bar{c}_j$ must have sufficiently large values, that is, $\bar{c}_j=c_j/\varepsilon$, where $\varepsilon>0$ is a small parameter which characterizes the elastic properties.

The moments of the elastic forces Q_j are generalized potential forces which correspond to the coordinates q_j. The mechanical system being considered will therefore have a potential energy

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$$\Pi^* = \frac{1}{2}\varepsilon^{-1}\sum_{j=2}^{n} c_j q_j^2 = \varepsilon^{-1}\Pi(q) \tag{1.1}$$

if we assume that there is a linear relationship between the stresses and strains. In the general case, the potential energy $\Pi(q)$ will have the form

$$\Pi^*(q) = \varepsilon^{-1}\Pi(q) = \varepsilon^{-1}\sum_{j=2}^{n}\Pi_j(q_j)$$

We shall assume that the functions $\Pi_j(q_j)$ are strongly convex functions of the variables q_j, which take a value of zero at the unique stationary point $q_j = 0$. In this case, the potential $\Pi(q)$ will be a positive-definite function of the variables $q_2, \ldots, q_n$ for which each of the equations

$$\partial\Pi/\partial q_j = \partial\Pi_j(q_j)/\partial q_j = 0 \tag{1.2}$$

will have a unique solution $q_j = 0$. Additionally, we shall assume that

$$\Pi(q) \to \infty \text{ when } \sum_{j=2}^{n} |q_j| \to \infty$$

In case (1.1), this property will be automatically satisfied.

The finite nature of the domain of elastic strains can be taken into account by considering finite segments with respect to the variables q_j, where the functions $\Pi_j(q_j)$ will be strongly convex. In this case, the property, established below, of the asymptotic stability of the trivial solution of the generating system as a whole is properly replaced by stability in the large, that is, within the limits of a bounded domain of initial deviations.

In the subsequent discussion, the fact that the potential energy $\Pi(q)$ is independent of the coordinate q_1, that is, the identity

$$\partial\Pi/\partial q_1 \equiv 0 \tag{1.3}$$

is important.

Since there are no other forces apart from elastic forces in the system, the system under consideration will be conservative when $M_1 \equiv 0$. Since, in the model being described, the control acts solely on the first link, the motion of the system in generalized coordinates $\{q_i\}$ $(i = 1, \ldots, n)$ will be described by the second-order Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = M_1\delta_{1i} - \varepsilon^{-1}\frac{\partial \Pi}{\partial q_i} \tag{1.4}$$

$$T = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(q)\dot{q}_i\dot{q}_k \triangleq \frac{1}{2}\dot{q}^{\mathrm{T}}A(q)\dot{q}$$

where δ_{1i} is the Kronecker delta and T is the kinetic energy, which is a positive-definite quadratic form of the generalized velocities.

With regard to the function $M_1(t)$, which defines the controlling moment and acts on the first link, we shall assume in the first pat of this paper that no constraints whatsoever are imposed on this function. The question of taking account of the natural constraints on the controlling moment

$$|M_1(t)| \leqslant h, \quad t \geqslant t_0, \quad h > 0 \tag{1.5}$$

is discussed in the concluding part of this paper.

Equations (1.4) belong to a class of singularly perturbed systems [1] which contain the small parameter accompanying the highest derivative. This reflects the existence in the system of motions with different frequencies [2]. Motions of the elastic part of the system correspond to rapid motions while the motion of the system as a solid body corresponds to a slow motion.

When $\varepsilon \to 0$, the differential equations (1.4) formally become the finite relationships

$$q_2 = \dots = q_n = 0 \tag{1.6}$$

which determine the rectilinear form of an absolutely rigid rod. However, the manifold (1.6), generally speaking, is not asymptotically stable. This fact additionally shows that, for the correctness of the discrete model (1.4), it is necessary to ensure the asymptotic stability of the manifold (1.6). Obviously, this can only be done by means of the appropriate introduction of a stabilizing controlling action. In order to determine the required control, we pass, in system (1.4), to a new (fast) time τ using the transformation

$$t = \varepsilon^{1/2}\tau \tag{1.7}$$

In accordance with (1.7), the equations of motion (1.4), written in matrix form, take the form of the Lagrange equations

$$\frac{d}{d\tau}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} = \varepsilon M_1(\varepsilon^{1/2}\tau)e^1 - \frac{\partial \Pi}{\partial q} \tag{1.8}$$

where e^1 is the first column of the unit matrix and the prime denotes a derivative with respect to τ. System (1.8) belongs to systems which contain a small parameter on the right-hand side.

If the product $\varepsilon M_1 \to 0$ as $\varepsilon \to 0$, which always holds in the case of bounded control, then the generating system corresponding to (1.8) will have the form

$$\frac{d}{d\tau}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} = -\frac{\partial \Pi}{\partial q} \tag{1.9}$$

In accordance with the small parameter method, the properties of the generating system define the motions of the perturbed system (1.8).

When $\Pi(q^0) > 0$ the conservative system (1.9) will execute non-decaying motions since there are no dissipative forces in the system. This shows that taking the limit as $\varepsilon \to 0$ from a singularly perturbed system to the manifold (1.6) is invalid, generally speaking, and, consequently, non-decaying elastic oscillations will be present in the resulting motion. The solution of control problems can therefore turn out to be generally impossible.

The arguments which have been presented show that the formation of a generating system, for which the manifold (1.6) is asymptotically stable, is a necessary element in the procedure for synthesizing the control of elastic systems. In order to do this, it is necessary to find a control law which ensures the introduction of a damping action into the generating system. It is shown below that such control laws exist and their analytic form is deduced.

2. CONTROL LAWS WHICH ENSURE THE STABILIZATION OF THE STEADY STATE OF A GENERATING SYSTEM

By (1.8), a control can only occur in the equations of a generating system when the controlling moment M_l contains a component which depends quadratically on the generalized velocity of the first link $\dot{q}_1$.

As an example of such a law, let us consider the function

$$M_1(t) = -\gamma\dot{q}_1^2\psi(\varepsilon^{1/2}\dot{q}_1) + u_1(t) \tag{2.1}$$

where $\psi(\zeta)$ is a smooth function which satisfies the condition $0<\psi(\zeta)\zeta$ when $\zeta \neq 0$ and $\psi(0)=0$ ($\psi(\zeta)=\operatorname{arctg}(\zeta)$ can serve as an example). By (1.8), the component $u_1(t)$ will not occur in the generating equations. It is intended for the control of the slow motion. The number $\gamma>0$ in (2.1) is introduced in order to ensure that the constraints on the control (1.5) are satisfied. This question is discussed in Section 3.

On replacing (1.7) and using the control law (2.1), the corresponding generating system will have the form

$$\frac{d}{d\tau}\frac{\partial T}{\partial q'} - \frac{\partial T}{\partial q} = -\gamma q_1'^2\psi(q_1')e^1 - \frac{\partial \Pi}{\partial q} \tag{2.2}$$

We will show that a decay of the motion along the generalized coordinates $q_2, \ldots, q_n$ occurs in system (2.2). In order to do this, we first note that Eqs (2.2) do not depend explicitly on q_1.

Actually, the kinetic energy of the multilink system being considered has the form

$$T = \frac{1}{2}\sum_{i,k=1}^{n}\gamma_{ik}\cos\left(\sum_{s=p}^{\nu} q_s\right)q_i'q_k'$$

$$\gamma_{ik} = l_i l_k\left(m_\nu + 2\sum_{r=\nu+1}^{n} m_r\right), \quad p=\min(i,k)+1, \quad \nu=\max(i,k) \quad i\neq k$$

$$\gamma_{kk} = l_k^2\left(\frac{1}{3}m_k + \sum_{r=k+1}^{n} m_r\right)$$

and does not depend explicitly on q_1 since $p \geqslant 2$.

Hence, on taking account of (1.3), it is possible to reduce the order of this system if one takes $q_1', q_2, q_2', \ldots, q_n, q_n'$ as the variables, that is, it is possible to introduce a dynamical system into the treatment, the state of which is described by a $(2n-1)$-vector $x=(q_1', q_2, q_2', \ldots, q_n, q_n')^\tau$ and the process of the exchange of states is determined by Eqs (2.2).

System (2.2) has a solution $x=0$ and it corresponds to the steady state and, furthermore, other steady states in the variables x do not exist.

Actually, all the terms on the left-hand side of system (2.2) contain as a factor one or two of the first derivatives of the generalized coordinates. In the state $x=0$, the left-hand sides of Eqs (2.2) are therefore equal to zero. It is obvious that, by virtue of (1.3) and the property of strong convexity of $\Pi(q)$, the right-hand sides of these equations vanish when $q'=0$.

Theorem. The state $x=0$ in a $(2n-1)$-dimensional space of states $x=(q_1', q_2, q_2', \ldots, q_n, q_n')^\tau$ is asymptotically stable as a whole with the state of the dynamical system (2.2).

Proof. We shall make use of the generalized Lyapunov theorem on asymptotic stability [3]. According to this theorem, the asymptotic stability of a state $x=0$ is proved, if a Lyapunov function $V(x)$ with the following properties is specified. The function $V(x)$: (1) takes a value of zero when $x=0$, (2) is positive definite when $x\neq 0$, (3) admits of an infinitely large lower limit, that is, $V(x)\to\infty$ when $|x|\to\infty$, (4) has a non-positive derivative $V'=dV/dt\leqslant 0$, and (5) the set $\{x: V'=0\}$ contains the point $x=0$ and does not contain other positive semitrajectories when $\tau\geqslant 0$.

As a Lyapunov function, let us consider the total energy of system (2.2)

$$V(x) = E(q,q') = T(q,q') + \Pi(q) = \frac{1}{2}\sum_{i,k=1}^{n} a_{ik}(q)q_i'q_k' + \frac{1}{2}\sum_{j=2}^{n} c_j q_j^2 \tag{2.3}$$

where a_{ik} are elements of the matrix of the kinetic energy coefficients $A(q)$. It follows from the expression for the kinetic energy T and condition (1.3) that the Lyapunov function (2.3) is defined and continuously differentiable in the $(2n-1)$-dimensional space $\{x\}$. Function (2.3) possesses the properties which have been enumerated above.

Let us show this. Property 1 is obvious. When account is taken of the positive definiteness of the kinetic energy T with respect to the variables q_i' and the above-mentioned properties of the potential $\Pi(q)$, we conclude that the function $V(x)$ is positive definite, that is, it possesses property 2. The function $V(x)$ admits of an infinitely large lower limit since, by our assumption, $\lim \Pi(q)=\infty$ when $|q_2|+\ldots+|q_n|\to\infty$ and $\lim T(q, q')=\infty$ when $|q'|\to\infty$, which immediately follows from the well-known kinetic energy inequality $T \geqslant a_0(q_1'^2+\ldots+q_n'^2)$, where a_0 is a positive number.

Since the derivative of the total energy of a mechanical system $dE/d\tau$ is equal to the power of the non-potential forces [4] then, in the case of system (2.2), we shall have

$$dE/d\tau = dV/d\tau = -\gamma q_1'^3 \leqslant \psi(q_1') \leqslant 0$$

that is, the Lyapunov function being considered possesses property 4. The following lemma establishes that it has property 5.

Lemma. If the first link remains fixed in a certain position during the motion of the system (2.2), that is, $q_1' \equiv 0$ then, during such motion, $q_2 = q_3 = \ldots = q_n \equiv 0$.

Proof. For convenience in describing system (2.2), we shall change to another system of generalized coordinates $\varphi = (\varphi_1, \ldots, \varphi_n)^{\mathrm{T}}$, where $\varphi_1, \ldots, \varphi_n$ are absolute angles, that is, the angles of deviation of the links from a certain fixed axis. The generalized coordinates q are expressed in terms of the coordinates φ by the equalities

$$q_1 = \varphi_1, \quad q_2 = \varphi_2 - \varphi_1, \quad q_3 = \varphi_3 - \varphi_2, \ldots, \quad q_n = \varphi_n - \varphi_{n-1}$$

The potential energy is therefore

$$\Pi = \tilde{\Pi}(\varphi_2 - \varphi_1, \varphi_3 - \varphi_2, \ldots, \varphi_n - \varphi_{n-1}) = \tilde{\Pi}(\varphi) = \sum_{j=2}^{n} \Pi_j(\varphi_j - \varphi_{j-1})$$

The expression for the kinetic energy is replaced by the expression

$$T = \frac{1}{2} \sum_{i,k=1}^{n} \gamma_{ik} \cos(\varphi_i - \varphi_k)\varphi_i'\varphi_k'$$

and, in the new variables φ, Eqs (2.2) take the form

$$\sum_{k=1}^{n} \gamma_{ik} \cos(\varphi_i - \varphi_k)\varphi_k'' + \sum_{k=1}^{n} \gamma_{ik} \sin(\varphi_i - \varphi_k)\varphi_k'^2 = -\gamma\varphi_1'^1\psi(\varphi_1')e^1 - \frac{\partial\tilde{\Pi}}{\partial\varphi_i} \tag{2.4}$$

Since $\varphi_1 \equiv 0$ according to the condition of the lemma the forces, generated by the motions of the remaining links, will only act on the first link. Since the first link is fixed, it is obvious that the reaction of the joint, which is applied to the first link at the point where it is joined to the second, will be directed along the axis of the first link. This means that, if the first joint were not firmly clamped and located at the end of an additional link (AL) and in this joint, as in the other joints, there was a spring with a rigidity c_0, then the motion of the links with numbers $2, \ldots, n$ under which link 1 remains fixed and would not have any effect on the additional link. Hence, the additional link, being fixed and directed along the axis of link 1 at the start of the motion, would, subject to the condition $\varphi_1 \equiv 0$, also maintain zero velocity during the motion of links $2, \ldots, n$.

In order to show this, let us consider a system of two links with numbers 0 and 1. Let them have lengths l_0 and l_1. Their position is described by the angles φ_0 and φ_1 of deviation of the links from the one and the same fixed direction, such as from the vertical, for example. If a force $P(\tau)$, which is directed for all t along the vertical, is applied to the free end of link 1, then the virtual work of the force $P(\tau)$ will be equal to

$$\delta A(\bar{P}) = -P(\tau)l_0 \sin\varphi_0 \delta\varphi_0 - P(\tau)l_1 \sin\varphi_1 \delta\varphi_1 = Q_0\delta\varphi_0 + Q_1\delta\varphi_1$$

where $Q_0 = -P(\tau)l_0 \sin\varphi_0$, $Q_1 = -P(\tau)l_1 \sin\varphi_1$ are the generalized forces in this system of two links. In the position $\varphi_0 = \varphi_1 = 0$, the generalized forces $Q_0 = Q_1 = 0$. It follows from the principle of possible displacements [4] that $\varphi_0 = \varphi_1 = 0$ is the equilibrium position of the two-link system and, consequently, if, at the initial instant of time, the system was in this position with zero velocity, then it would not leave this position with the passage of time.

Hence, subject to the condition $\varphi_1 \equiv 0$, the addition of a further link to a system of n links described by Eqs (2.4) directed along the axis of link 1, does not alter the motion of links 2, . . . , n. Let us assign the number zero to the additional link. The equations of motion of the system with an additional link are analogous to Eqs (2.4) with the sole difference that the indices i and k take values not from one to n, as in (2.4), but from zero to n. The potential energy $\tilde{\Pi}$ of a system with an additional link contains one more term $(\Pi_1(\varphi_1 - \varphi_0))$ than the potential energy of system (2.4).

The condition that the links with numbers zero and one are fixed is expressed by the identities

$$\varphi_0 = \varphi_1 \equiv 0, \quad \varphi_0' = \varphi_1' \equiv 0, \quad \varphi_0'' = \varphi_1'' \equiv 0 \tag{2.5}$$

Let us assume that the remaining links execute a certain motion. It follows from (2.5) that

$$\frac{\partial\tilde{\Pi}}{\partial\varphi_0} = \frac{\partial\tilde{\Pi}_1(\varphi_1 - \varphi_0)}{\partial\varphi_0} = 0$$

$$\frac{\partial\tilde{\Pi}}{\partial\varphi_1} = \frac{\partial\tilde{\Pi}_1(\varphi_1 - \varphi_0)}{\partial\varphi_1} + \frac{\partial\tilde{\Pi}_2(\varphi_2 - \varphi_1)}{\partial\varphi_1} = \frac{\partial\tilde{\Pi}_2(\varphi_2 - \varphi_1)}{\partial\varphi_1}$$

When account is taken of this, the first two equations of the system with an additional link will have the form

$$\sum_{g=2}^{n} \gamma_{0g}\varphi_g''\cos\varphi_g - \sum_{g=2}^{n} \gamma_{0g}\varphi_g'^2 \sin\varphi_g = 0 \tag{2.6}$$

$$\sum_{g=2}^{n} \gamma_{1g}\varphi_g''\cos\varphi_g - \sum_{g=2}^{n} \gamma_{1g}\varphi_g'^2 \sin\varphi_g = \frac{\partial\tilde{\Pi}_2(\varphi_2 - \varphi_1)}{\partial\varphi_1} \tag{2.7}$$

In these equations, summation is carried when $g = 2, \ldots, n$ since, when $g = 0$ or 1, the corresponding terms are equal to zero by virtue of conditions (2.5). Since $\gamma_{0g} = (l_0/l_1)\gamma_{1g}$ when $g \geq 2$, the left-hand sides of Eqs (2.6) and (2.7) differ by a constant factor l_0/l_1. It therefore follows from them that $\partial\tilde{\Pi}_2(\varphi_2 - \varphi_1)/\partial\varphi_1 \equiv 0$ and from this the identity $\varphi_2 = \varphi_1 \equiv 0$ follows by virtue of (1.2). On applying analogous arguments successively to the remaining equations of the system with an additional link, we obtain the identity $\varphi_3 \equiv 0, \ldots \varphi_n \equiv 0$. As was pointed out above, the motion with respect to the coordinates $\varphi_2, \ldots, \varphi_n$, which is described by the equations of a system with an additional link and subject to condition (2.5), is identical to the motion along these coordinates described by Eqs (2.4) subject to the condition $\varphi_1 \equiv 0$. Hence, the identities $\varphi_2 \equiv 0, \ldots, \varphi_n \equiv 0$, obtained for a system with an additional link subject to condition (2.5), are also satisfied in the case of system (2.4) subject to the condition $\varphi_1 \equiv 0$. The lemma is proved.

All the conditions of the generalized Lyapunov theorem on asymptotic stability are therefore satisfied. The theorem is proved.

It follows from the theorem that the control (2.1) ensures the asymptotic stability of the manifold (1.6) as a whole in the case of a generating system (2.2). System (1.8) with the control (2.1) differs from the generating system (2.2) by the presence of a term $\varepsilon u_1(\varepsilon^{1/2}\tau)e^1$ on the right-hand side.

By virtue of a theorem due to Malkin [5], the state $x = 0$ $(x = \{q_1', q_2, q_2', \ldots, q_n, q_n'\}^{\mathrm{T}})$ in the case of a bounded control $u_1(t)$ and sufficiently small ε, will be stable when there are constantly acting perturbations. This means that the energy of the elastic forces will remain small, if it was small at the initial instant of time. The possibility arises of realizing slow programmed motions with respect to the variable q_1. For this purpose, one can use an expansion of the solution of the above-mentioned system with respect to a small parameter. The control u_1 will occur in the equation of the first approximation, the solution of which will also determine the motion in accordance with the purpose of the control.

Note that the effect of the damping of the elastic oscillations using a controlling action is of a non-linear nature. If, instead of the control (2.1), one adopts a control which depends linearly on $\dot{q}_1$, then the corresponding generating system will not contain damping forces, that is, it will be conservative. This, as has already been mentioned above, implies that the energy of the elastic forces is conserved and non-decaying motions will exist in the generating system.

3. CONDITION OF THE BOUNDEDNESS OF THE CONTROL

Let us consider the question associated with the constraint on the control in the form of the inequality (1.5). It follows from expression (2.1) that formally the control must be unbounded if one bears in mind that the parameter tends to zero. It follows from the theorem which has been established that the control will ensure the damping of elastic vibrations of as high a frequency as may be desired. Hence, if the frequencies of the elastic vibrations are bounded, which actually occurs, then the magnitude of h in (1.5), in accordance with (2.1), will be of the order of the square of the maximum frequency. This fact shows that, using a bounded control, it is impossible to ensure the damping of elastic vibrations in the remote part of the spectrum. It is clear from physical considerations that the amplitude of such vibrations will be quite small so that the shape of an elastic rod will differ only slightly from the rectilinear shape of an absolutely rigid rod.

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